Universality of k^{-1} noise, the enstrophy cascade, and the large-scale atmospheric spectrum

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Direct numerical simulations of two-dimensional (2D) incompressible Navier-Stokes turbulence can model large-scale atmospheric dynamics when driving and dissipation cover wide ranges of length scales. Natural assumptions for the 2D energy balance lead to the robust k^{-1} vorticity spectrum ($k = 2\pi$, length scale) observed in the atmosphere. Scaling in k space is related to hyperbolicity of large-scale 2D flow in physical space. The mechanism of this scaling has similarities with a recent model of dissipative self-organized criticality.

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I. INTRODUCTION

The kinetic energy spectrum of atmospheric motion scales like k^{-3} with the spatial wave number k, for both longitudinally and azimuthally averaged spectra, at lengthscales $(0.3–3)\times 10^3$ km [1,2]. Structure functions and flux rates evaluated from a large set of aircraft flight data show [2] that this is consistent with an enstrophy cascade in two-dimensional turbulence (2DT). Kraichnan [3] has shown that an inertial range dominated by such a cascade should display a k^{-3} energy spectrum, or equivalently a k^{-1} enstrophy spectrum, where the enstrophy density $E = \omega^2/2$ and ω is the scalar vorticity in a two-dimensional flow. Kraichnan's original idea was of a narrow band forcing with the enstrophy cascade occurring at wave numbers larger than those of forcing, and with a $k^{-5/3}$ spectrum determined by inverse energy cascade at smaller wave numbers. But the atmospheric energy is known [2] to scale in exactly the opposite way, with k^{-3} at smaller wave numbers and $k^{-5/3}$ at larger ones. Moreover, the assumption of an inertial range where both forcing and damping can be considered negligibly small in comparison with inertial effects is rather questionable in the case of atmospheric motion, which is vigorously forced over all lengthscales involved in it.

A. Present results

In Sec. III it is argued on the basis of observations from extensive direct numerical simulations (DNS) of 2DT that a k^{-1} enstrophy range obtains robustly, even when the forcing includes or is included in the scaling range of wave numbers. Such a range must still be ''nearly inertial,'' in the sense that any forcing effects have to be small, but not at all negligible, when compared with the amount of enstrophy in the modes on which they act. The equations and forcing scheme for the present DNS are described in Sec. II. The novelty is the forcing wave number band.

To explain this robustness and to show that it still represents an enstrophy cascade to larger wave numbers as Lindborg [2] found from his data, the dynamical mechanism underlying the scaling is identified as the stretching of material volumes, which correlate strongly with vorticity in high-Reynolds-number two-dimensional turbulence. The idea that stretching is the generator of turbulence is very old; it is reviewed briefly below. The important point in the present case is that a *linear* theory is sufficient, without any linearizing assumptions to be made. This seems unusual in turbulence theory. All relevant theoretical arguments found in the literature are collected and supplemented in several aspects in Sec. IV. A review of 2D DNS articles in Sec. VI offers further justification of the assumption of linear interaction between large-scale strain and small-scale vorticity, and suggests that the k^{-3} scaling of the enstrophy cascade toward small scales and the $k^{-5/3}$ scaling of the energy cascade from those same small scales can coexist as if they were linearly independent fluxes. Such a picture seems even more unusual in turbulence theory, despite the long recognized ''nonlocality'' (in wave number space, or ''locality'' in physical space) of 2DT transfer processes.

The ''universality of k^{-1} noise'' has recently attracted much attention in the theory of dissipative self-organized critical systems (DSOCSs). A close correspondence is established in Sec. V between the linear aspects in the evolution of spectra in 2DT and in a recent more abstract DSOC model system, which also displays robust k^{-1} scaling. It can therefore be said that the atmosphere of our planet is yet another example of that universality.

B. Relation to previous results

The two key points in the explanation of k^{-1} scaling are (1) nearly linear dynamics of small-scale vorticity with respect to advection by 2DT velocity and (2) dominance of layerlike structures at small spatial scales. The latter is illustrated in Fig. 1, which is a vorticity snapshot from one of the present DNS. The importance of filamentation in 2DT was noticed long ago in DNS of atmospheric dynamics models [1]. It has been recognized as the mechanism of enstrophy cascade by Batchelor [4] and Kraichnan [3]. (They have considered it important in 3D turbulence as well [5,6].) DNS of 2DT [7,8] at higher-resolution (4096^2) than used here, but forced at the lowest few wave numbers (which is the currently standard setup), have produced a k^{-1} enstrophy spectrum, over roughly one octave of wave numbers. Gotoh [8] noted the massive presence of layerlike vorticity structures.

Both 2DT theory and DNS related to enstrophy cascade and the k^{-1} range assume forcing over a band of wave numbers $k_F = O(1)$ where the smallest resolved wave number in

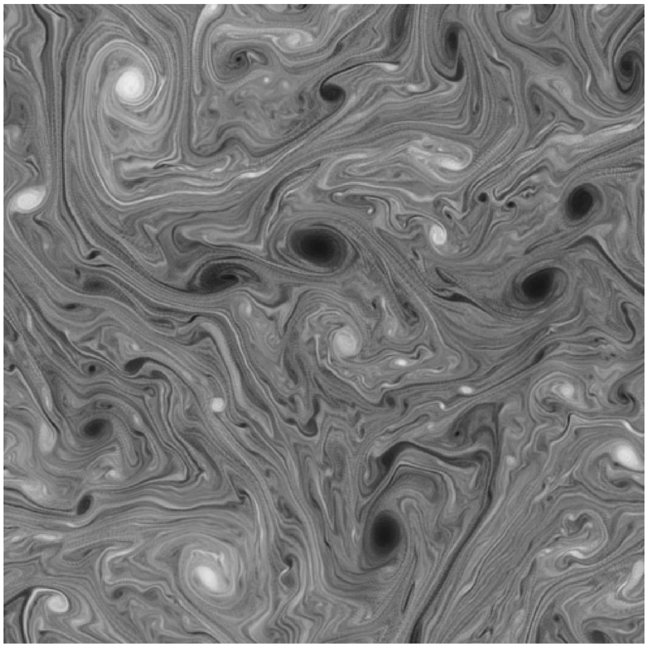

FIG. 1. Instantaneous vorticity in run M (see the middle bold line in Fig. 2 and Table I).

DNS $k_{\min}=1$ and dissipative-scale wave number $k_d \gg 1$. The problem with this classical forcing is that it insists on the existence of an inertial range of wave numbers, where forcing and damping can be neglected. So far it has been shown that a k^{-1} scaling obtains only under such conditions. As already noted, these restrictions are both too strong and too unrealistic for the atmospheric case.

The classical theory of isotropic turbulence is presented entirely in Fourier space, so it can reflect the importance of coherent structures in physical space only through specific scalings like the k^{-1} one studied here. Batchelor's three-dimensional passive scalar scaling theory [5] and later Kraichnan's theory of random advection of a passive scalar [9] in arbitrary dimension, which will be adapted here to describe the 2D vorticity scaling, come in a representation that employs both spectral and physical space, with statistical arguments implied in the transition between them. At the basis of these theories is the concept of filamentary small-scale structure which results from average stretching. Batchelor [5] assumed nonzero average large-scale strain to derive the k^{-1} scaling for a passive scalar spectrum in the convective-dissipative range, while Kraichnan showed that stretching occurs on the average even if the average strain vanishes. We find that in 2D Kraichnan's result can be obtained even without assuming isotropy, so it extends to, e.g., β-plane turbulence.

Even if it vanishes on the average, the velocity strain in 2DT divides the flow into regions of elliptic and hyperbolic large-scale flow. A comment on the role of the former is given in Sec. VI. The latter has special significance in 2DT: A converging large-scale flow induces strong vorticity gradients (filamentation); the gradients then induce large-scale shear. This leads us to the oldest models of frontogenesis [10], which consider instabilities in a 2D simple shear or stagnation flow. Such large-scale motions then sustain the otherwise three-dimensional and spatially localized mechanism of frontogenesis. A set of exact solutions [11] for vorticity gradients driven by large-scale stagnation flow allows more rigorous modeling of filaments in the present "randomly advected vorticity" variant of Kraichnan's theory.

II. DYNAMICAL MODEL

Results reported here are for quasigeostrophic (QG) β-plane scalar vorticity dynamics. In the QG approximation [12] the large-scale height-averaged motions are small deviations from a global balance between atmospheric pressure and the Coriolis force due to rotation of the Earth. When the largest scales of interest are of order 10^3 km, the Coriolis term can be linearized into a β-term. A random force over some *broad* band of spatial frequencies crudely models the cumulative effect of a multitude of different factors, e.g., convection, gravity-wave instabilities, topography. A linear damping term models the combined effect of all dissipative mechanisms effective at the numerically resolved scales, e.g., boundary layer friction or radiation. The nonlinear dynamics immediately generates scales below the resolvable ones; this flux is disposed of by an arbitrarily chosen hyperviscous term. The dynamics over a domain periodic in each of the spatial coordinates x_1 and x_2 is governed by

$$[\partial_t + \nu(-\nabla^2) + v_1\partial_1 + v_2\partial_2]\omega = \beta v_2 + f, \tag{1}$$

$$(v_1, v_2) = (\partial_2, -\partial_1)(-\nabla^2)^{-1}\omega, \tag{2}$$

$$\partial_t f(x_1, x_2, t) = -\gamma f + g(x_1, x_2, t), \tag{3}$$

where $\nabla^2 = \partial_1^2 + \partial_2^2$; the damping model $\nu(L) = \nu_0 + \nu_m L^m$ has constant $\nu_0, \nu_m > 0$; the order of (hyper)viscosity $m \geq 1$ is integer. The random stirring force f follows a popular model [13]: The Langevin equation (3) with memory due to the damping factor $\gamma > 0$ is driven by a Gaussian white noise g. When integrating numerically with time step dt, relevant parameters are the forcing range bounds $k_b < k_e$, a normalized forcing amplitude $g_0 = \langle g \rangle / \gamma$, and a memory factor $g_1 = 1 - dt\,\gamma$.

The system (1)–(3) is a rather general example of an *incompressible* Navier-Stokes-like *two-dimensional* dynamical system. It reproduces some key features of global-scale atmospheric dynamics: (1) dominance at the largest scales of zonal flow [14] due to the β-term, (2) filamentation, and (3) the k^{-3} energy spectrum scaling, which is our subject here.

III. NUMERICAL RESULTS

We now argue that, in order to predict an enstrophy (or passive scalar) cascade and its direction one needs to verify only the following conditions: (1) dominance of kinetic energy *content* over its input/output rate, and (2) conservation of the scalar on each fluid particle by the nondissipative and unforced dynamics. Clearly, condition (2) is satisfied by Eq. (1); it is specific for 2D systems. Condition (1) is weaker than requiring an inertial range; it allows forcing and damping *anywhere* in physical and wave number space, as long as redistribution of fluxes by inertial effects is fast.

Our main numerical result is the observation from DNS that a nearly statistically steady state with well-defined k^{-1} enstrophy spectrum obtains *always* when condition (1) holds.

TABLE I. Parameter ranges used in DNS.

	β	ν_0	ν_m	m	g_0	g_1	k_b	k_e
Minimum	0	10^{-4}	10^{-1}	1	10^{-4}	0	1	4
Maximum	10	10^{-1}	10^{3}	12	0.4	0.99	220	250
Run *A*	0	2×10^{-3}	1	10	10^{-2}	0.83	112	225
Run *M*	0	10^{-1}	40	10	10^{-1}	0.99	112	125
Run *S*	0	10^{-4}	10	10	10^{-1}	0.83	220	225

Here ''well defined'' means up to two decades in *k*, for 512^2 spatial resolution ([7,8] report smaller ranges at higher resolution). Technically, condition (1) implies two restrictions: (1) that the initial enstrophy spectrum falls off at $k \gg 1$ faster than k^{-1}, and (2) the forcing amplitude $\langle g \rangle$ (which is the same at any wave number for the scheme used here) for $\langle f \rangle$ is small compared with the enstrophy content at $k = O(1)$.

Independence of forcing details and of the requirement for an inertial range has been verified by obtaining the same scaling, after transients of nearly the same duration, from a large variety of run parameters: the damping model and forcing ''memory time'' were varied, and forcing was applied in one or *two* bands, with amplitudes selected nearly at will, but independent of *k* in each band. Different forcing bands were used: (1) only at low wave numbers $k = O(1)$ (classical case), or (2) only at high wave numbers $k = O(k_d)$,[1] or (3) only in a narrow band of wave numbers $1 \ll k \ll k_d$ ''in the middle'' of the resolved range, or (4) over nearly the *whole range* of resolved scales (this case is qualitatively closest to modeling atmospheric dynamics).

Initial vorticity fields were randomly generated to match one of a few prescribed initial spectra, all dominated by large-scale motions. A substantial number of runs was needed to explore the portion of parametric space specified in the upper half of Table I. Details about this parametric study will be omitted, since only the qualitative result is of interest here: The k^{-1} scaling results in all cases in which both forcing and damping remain below some empirical threshold, which is close to the upper bounds on ν and g_0 in Table I when initial energies are $O(1)$.

To illustrate the qualitative independence of the final outcome from the forcing details, the spectral evolution for a few representative runs is shown in Fig. 2. Forcing was applied at different wave number bands (*S*: extreme small-scale narrow band, *M*: moderately small-scale narrow band, *A*: broadband). The classical forcing case is omitted. These three runs were started from the same initial field, with energy concentrated around $k = 4$, and advanced with the same kind of hyperviscosity. Table I lists the parameter values.

The universality of k^{-1} scaling has now to be explained without reference to an inertial range, because $\nu_0 > 0$ for all *k*, and DNS have shown that the location and width of the forcing band is of no relevance and may overlap with the

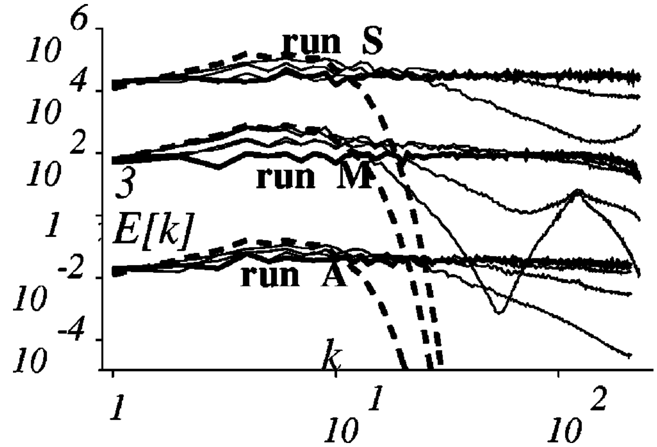

FIG. 2. Evolution of enstrophy spectrum in runs *S* (top), *M* (middle), and *A* (bottom). Thick lines: initial time (dotted) and latest record (solid). Data are compensated (horizontal lines correspond to k^{-1}) and vertically shifted by arbitrary rescaling factors: $10^{-3}(S), 7.5(M), 10^{3}(A)$.

scaling range. The idea that small-scale vorticity is advected in 2D similarly to a *passive scalar* goes back at least to Taylor (for a recent review and original numerical employment of this idea, see [15]). For the cases of nearly inviscid vortex merger, decaying turbulence, and wave numbers above the forcing range in a narrow band-forced 2DT with strong input, i.e., in violation of restriction (2), detailed numerical verification has been given [16]. Our present results suggest that in a statistically quasi-steady-state satisfying restrictions (1) and (2), vorticity is advected in that way at *all* wave numbers, even if forcing is *not* narrow band.

IV. ADVECTION OF SMALL-SCALE VORTICITY

Here we adapt Kraichnan's theory of random advection of passive scalars [9] to the advection by 2DT of any scalar conserved on fluid elements. In the inviscid limit, vorticity (not only its small-scale part) is such a scalar. Kraichnan considers advection by a white-noise velocity field of arbitrary (integer) dimension $D \geq 2$, and derives the k^{-1} scaling on the basis of several assumptions, but without reference to nonzero average strain. He first shows that *isotropy* together with *incompressibility* lead to positive average stretching of material lines, although the strain tensor vanishes on the average. Therefore, material elements are stretched, on the average, into ''thin sheets'' of dimension $D-1$. He then introduces a convenient but arbitrary model of the cross-sectional profile of the scalar gradient across those sheets, choosing them to correspond to an input centered at a fixed wave number, and then studies the time evolution of his model. For an ensemble of such sheets, and using the time-dependent average and dispersion of the line stretching rate, he then computes the spectrum in a way similar to Batchelor's [5]. Here we have to relax the requirements that the velocity be isotropic, that it be white-noise or δ correlated in time (and the related time dependence of line stretching statistics), and that vorticity input have a fixed characteristic length scale. The arbitrary model of filamentary structure is to be replaced by one that is a solution to the dynamical equations.

[1]The high-*k* forcing scheme is traditionally used to produce the $k^{-5/3}$ energy scaling of the inverse 2DT cascade, and not the k^{-3} energy scaling of the forward enstrophy cascade as done here. The crucial difference is that, to simulate an inverse cascade, an initial field is picked up with ''insufficient'' energy or none at all.

A. Average stretching by random advection

Kraichnan's derivation employs in an essential way a white-noise assumption on the velocity field, which is clearly irrelevant as a model of two-dimensional motion, whose spectrum is dominated by large-scale modes. His comment on this point is a reference to a proof for more general statistics, given earlier by Cocke [17]. The proof is only for $D=3$; it uses incompressibility and isotropy. Below follows a sketch of its 2DT analog needed here.

A material line element $\delta l(0)$, when advected for a time interval t by an incompressible 2DT flow, when viewed in a Lagrangian frame moving with its center, appears mapped into $\delta l(t)=U\delta l(0)$. Here $\det(U)=1$ and $W=U^T U$ is a symmetric matrix with eigenvalues $w^{-1},w>0$. Transition from the Lagrangian frame to the orthogonal eigenvector frame of W is a solid rotation, determined in 2D by a single angle θ. The analog of Cocke's Eq. (5) for 2D is $l^2=|\delta l(t)|^2/|\delta l(0)|^2=(\cos\theta)^2 w+(\sin\theta)^2 w^{-1}$. The *logarithmic stretching rate* $\tilde{\mu}=-\log(l^2)/2$ can be bounded by noting that $-\log$ is a convex function: $2\tilde{\mu}\geqslant(\cos\theta)^2\log(w^{-1})+(\sin\theta)^2\log(w)$; equality implies $w=1$.

Now consider an *ensemble* of line elements. Since any $\delta l(0)$ is equally probable with its copy rotated by $\pi/2$, $\langle\log w\rangle=\langle\log(w^{-1})\rangle=0$. Thus $\lambda=\langle\tilde{\mu}\rangle>0$ when the standard deviation $\sigma(w)>0$. For $D\geqslant 3$ there are several w's. In order to have the lower bound on $\tilde{\mu}$ vanish on the average, one integrates with respect to several θ's, then averages, then assumes isotropy to get all $\langle\log w\rangle$'s equal. For $D=2$ there is one w and isotropy is not required.

This is corroborated by the well-known fact that β-plane turbulence also produces k^{-1}scaling. In the present series of DNS, results similar to those in Fig. 2 were also obtained for a β effect comparable to the total initial energy, again for different forcing bands and other run parameters.

B. Thin sheets

Kraichnan's next assumption [9] is that the characteristic scales of the spatial variation of average strain are much larger than the average thickness of the vorticity sheets. Therefore a small-scale fluctuation with scalar variance spectrum [cf. his Eqs. (3.4) and (3.5)] given at time $t=0$ by $E_p(a_p,b_0;k,0)=a_p b_0 f_p(kb_0)$, with characteristic cross-sectional scale b_0 and amplitude a_p determined by the normalization $\int_0^\infty f_p(k)dk=1$, will keep its shape nearly unchanged while adjusting its characteristic length $b(t)=l(t)b_0$ according to the experienced strain history: $E_p(k,t)=a_p b f_p(kb)$. For an infinitesimal material line element $\delta l(t)$ centered at $x(t)$ and moving in a strain field $S(x,t)$, that history is represented by $\tilde{\mu}(t)=\int_0^t S(x(t),t)dt$. So far, only incompressibility and the separation of scales between strain and vorticity were assumed, and only a single ''thin sheet'' was considered.

To estimate correctly the k^{-1}scaling range in Eqs. (5) and (6) below, one needs to know what are the actual typical cross-sectional profiles. Kraichnan assumed the form of sheets to be *sinusoidal*. Although the scalar advection problem is linear and so any profile is admissible as input, it seems natural to focus on filaments having their cross sections spatially localized and compatible with the flow dynamics. The filamentation of vortex blobs continues until they are so thin that further stretching is countered by viscosity. Their ''natural'' shape can be approximated by dissipative-scale solutions similar to the Burgers vortex tube and layer solutions, which are popular models for filamentary (tubelike and layerlike) vorticity structures abundant in three-dimensional turbulence.

Here follows a summary of the derivation in [11]. The first step is to find *steady state* solutions describing the exact balance between (a) large-scale strain, modeled by a stagnation flow, $(u,v)=S(-x,y)$ in local coordinates, and (b) dissipation $\nu(-\nabla^2)$. In the solutions, the profile of sheets is a function only of their cross-sectional coordinate, aligned with the direction x of compressional strain. In 2DT, steady state *vorticity layers* cannot be found. A ''stretched vector'' field is needed as the analog of 3D vorticity. This turns out to be the *Laplacian of velocity*. Steady profiles $h(x)$ for that component of the field which is along the direction of positive strain are exact analogs of the Burgers vorticity layer solutions in 3D. For Newtonian viscosity, $h(x)$ is Gaussian. For $m>1$ it is even more localized in Fourier space, and oscillatory (but not sinusoidal) in x space, with algebraic decay at $|x|\to\infty$.

In the next step, these steady profiles $h[\nu](x)$ are used to construct self-similar solutions, which model the process of stretching as adjustment, exponential in time, of their amplitude and length scale until their cross-sectional scale saturates at $O(k_d^{-1})$ where $k_d^{-2}=\nu/\lambda$. Clearly, such solutions can be disrupted by a dissipative-scale forcing, as can be the case with atmospheric dynamics. Moreover, the time to reach a steady state depends on the kind of small-scale dissipation. Therefore, no universal behavior can be expected for dissipative-scale dynamics. In any case, the vorticity is stretched into stripes of various thickness, with sharp boundaries of thickness $O(k_d^{-1})$.

C. Ensembles

Following Kraichnan [9] again, consider an ensemble of filaments created with the *same shape* at the *same time*, but with varying a and b_0, and at different $\vec{x}(0)$. Averaging over such an ensemble requires knowledge about the distributions of the initial parameters and of strain histories. In the case of forcing in a narrow band and with fixed amplitude, which is relevant to some of the DNS reported above, one may approximately assume uniform distributions over narrow intervals for both a and b_0. The distribution density $\tilde{P}(\tilde{\mu})$ is unknown; Kraichnan postulated it to be normal. Recall[2] that $\int\tilde{P}(\tilde{\mu})d\tilde{\mu}=\lambda>0$, and rescale $\tilde{\mu}=\lambda\mu$, so $\tilde{P}\to P$. If sheets in the ensemble are statistically independent, their total contribution to the spectrum is approximately

$$\langle a_p b_0\rangle_E \sigma_E(a_p)\sigma_E(b_0)\langle E_p(1,1;k)\rangle_S .$$

[2]Steady state turbulence implies time-independent $P(\mu)$. Kraichnan's [9] focus is the evolution of single material line statistics, so a time dependence of $\langle\mu\rangle$ appears in his analysis; his additional white-noise assumption gives $\langle\mu_W\rangle=t\lambda_0$ and $\sigma(\mu_W)=t^{1/2}\sigma_0$ with constants $\lambda_0,\sigma_0>0$.

Subscripts E,S denote averaging respectively over the ensemble of sheets and over the total stretching of material lines $l=e^{\lambda\mu}$. The latter has its probability density completely determined by the (unknown, centered) density P of μ: $P_S(l)=P(\mu-\lambda)(dl/d\mu)$. Denoting $b_p=\langle b_0\rangle_E$, $\langle E_p(1,1;k)\rangle_S\approx\int e^{\lambda\mu}f_p(e^{\lambda\mu}kb_p)P(\mu-\lambda)d\mu$. It remains to average over the distribution of admissible shapes f_p, and in the case of a *broadband* forcing, over input scales b_p weighted by corresponding forcing amplitudes. Irrespective of the probability measures arising in these two integrations over distributions of filament parameters, there can be found an approximate dependence H on the self-similar profile shapes f_p and on the parametrization of sheets based on f_p, such that

$$\langle E(k)\rangle\approx\int e^{\lambda\mu}H(k\,e^{\lambda\mu})P(\mu-\lambda)d\mu. \tag{4}$$

Kraichnan avoided in his analysis the two averaging steps discussed here, by insisting on a single-k forcing (fixed b_0), and on a fixed (sinusoidal) form of f_p.

The prevailing population of f_p's would consist of ''thin sheets,'' well approximated by the Burgers-vortex-layer type of solutions discussed already. Each of them, and therefore H as well, will have a Gaussian or faster falloff in k. With $\zeta=ke^{\lambda\mu}$ and $\Delta=\log(\zeta)-\log(k)-\lambda^2$,

$$\langle F(k)\rangle\approx(\lambda k)^{-1}\left(\int_0^\infty P(\Delta(\zeta)/\lambda)H(\zeta)d\zeta\right). \tag{5}$$

The integral above may be approximated by $P(0)\int_0^\infty H(\zeta)d\zeta$ if $|\Delta(\eta_H k_d)|\ll\lambda\sigma(\mu)$, where $\eta_H=O(1)$ is large enough, say $\eta_H=5$, in order that both $\int_{\eta_H}^\infty\hat{h}(\zeta)d\zeta$ and $\int_0^{1/\eta_H}\hat{h}(\zeta)d\zeta$ are negligible when $\hat{h}(\zeta)$ is the Fourier transform of any of the steady profiles $h(x)$. Then the integral in Eq. (5) has negligible k dependence, and a k^{-1} scaling obtains, approximately over the wave number range

$$k_He^{-\lambda\sigma(\mu)}\ll k\ll k_He^{\lambda\sigma(\mu)},\qquad k_H=e^{-\lambda^2}k_d(\lambda,\nu_m). \tag{6}$$

Evaluation of the average and dispersion of the logarithmic stretching of material lines in 2DT is beyond the scope of the present article. Qualitatively, $\sigma(\mu)$ grows if the energy-containing range $[k=O(k_E)$, where k_E is the peak of the spectrum] of the velocity spectrum shifts to larger lengths. One may conjecture a scaling law $\log(\sigma)\propto\log(k_E/k_d)$, expecting $\sigma(\mu)\gg1$ in 2DT.

V. RELATION TO A DSOCS MODEL

The QG dynamical model system studied so far was seen to produce a k^{-1} spatial spectrum for the advanced field ω under ''mild'' forcing. This behavior can now be compared with a forced DSOCS model system [18] that also produces k^{-1} spectra under a rather general setup. The evolution rules for the one-dimensional version of that model can be recast as follows:

$$a(\mu,b+db,0)=\epsilon\delta(\mu)f(b)db+\lim_{t\to\infty}a(\mu,b,t), \tag{7}$$

$$\frac{a(\mu,b,t+dt)-a(\mu,b,t)}{dt}=-\nu_E\,\Pi a(\mu,b,t)+\frac{\Pi a(\mu+d\mu,b,t)-2\Pi a(\mu,b,t)+\Pi a(\mu-d\mu,b,t)}{2/(1-\nu_E)}. \tag{8}$$

Here $\delta(0)=1$ but $\delta(\mu)=0$ for $\mu\geqslant1$; $\Pi a=a$ if $0\leqslant a<\epsilon_0$ but $\Pi a=0$ otherwise; $\epsilon_0\ll\epsilon\ll1$ and $\nu_E<1$ are positive constants. The random force f was chosen uniformly distributed over $[0,1)$; it is effective only at one end of the ''lattice'' $\mu=0,1,2,\ldots,M\gg1$, so there is a predefined ''direction of propagation'' for $\hat{E}(b)=\Sigma_\mu a(\mu,b,\infty)$ called ''energy'' in the model. Advancement in μ and then in b is effected in discrete steps, fixed at $d\mu=db=1$. Fourier transform of $a(\mu,b,0)$ in b gives spectral densities $E(\mu,k)$. Numerical experiments [18] showed that all cases with $\nu_E\geqslant0.01$ were found to scale as $E(k)\sim k^{-1}$. The explanation offered in [18] was based on the observation that the spectrum for $\mu\gg1$ and for any fixed μ scales like $E(\mu,k)\approx e^{\mu\lambda}H(k\,e^{\mu\lambda})$ with constant $\lambda>0$. Note that this is the form of the function whose expectation is given by Eq. (4). No explanation for such an asymptotic form was given, only fits for λ from simulation data [18]. Assuming that $\Sigma_\mu E$ is dominated by these $\mu\gg1$ contributions, $E(k)\approx\int E(\mu,k)d\mu\approx(k\lambda)^{-1}\int H(\zeta)d\zeta$. Our Eqs. (5) and (6) imply the same, when $P(0)=1$. Enstrophy in our case is clearly the analog of DSOCS ''energy.''

Relaxation by Eq. (7) can be interpreted as straining of material elements or vorticity patches (blobs). Blobs can be assumed to appear initially with no preferred shape and orientation, and with average aspect ratio close to 1. If $\nu_0>0$, the vorticity associated with each blob decays. As folding occurs near stagnation points, filaments can be lumped, for statistical purposes, into blobs of larger characteristic size and smaller aspect ratio. Thus well-stretched blobs ''vanish'' at a rate $\sim\epsilon$ while unstretched blobs are ''generated'' at a similar rate at $\mu=0$. Equation (7) remains formally unchanged after Fourier transformation $(b\to k)$. When $d\mu,dt\to0$ and $(1-\nu_E)\propto\mu^{1/2}$, and if the effect of Π can be neglected, it represents a linearly damped diffusion process. This is a direct analogy of Kraichnan's stochastic diffusion of material elements (blobs) ''created'' with characteristic length scale b. Π prevents diffusion and decay of blobs with very weak vorticity. This means sustaining a weak multiplicative forcing of 2DT, i.e., modeling the weak background turbulence which does not belong to coherent vortices or vorticity layers. DNS with weak broadband forcing have shown that it is sufficient to support k^{-1} scaling in 2DT. In summary, a 2DT analog of ν_E should lump together three small parameters: $1/\sigma(\mu), \nu_0$, and ϵ.

A naive interpretation of Eq. (7), admissible only when 2DT forcing is narrow band, is to use b as a label of a ''slow time,'' as originally meant in the DSOC model, and not for injection length scale. This can be associated with evolution of spectra, as in Fig. 2, which occurs on a time scale slow compared with the exponential thinning of layers. Not only the restriction on forcing range, but also relating temporal to spatial spectra, are problems with this interpretation. Whether $b \gg 1$ means time or length, it can be associated with a large energy available in $a(b)$. In the $\mu \gg 1$ range, assumed decisive for the spectrum, no energy is added and $\Pi a = 0$ becomes important; a nearly b-independent (''quasi-steady'') state can be expected there. Changes enforced by Eq. (7) will therefore be relatively small when $b, \mu \gg 1$—an assumption strengthened by $\epsilon \ll 1$, and analogous to condition (1) in the 2DT discussion. A linear response to forcing, $a(\mu,b,\infty) - a(\mu,b,0) = A(\mu)db$, is expected in that parameter range; it corresponds to the approximation by passive scalar dynamics of the evolution of vorticity blobs in 2D. Adding $-a(\mu,b+db,\infty)$ on both sides of Eq. 7, then taking $db \to 0$ and Fourier transforming, leads to $ikE(\mu,k) = \hat{A}(\mu)$. Thus, k^{-1} scaling may follow from Eq. (7) without necessitating the particular form of Eq. (8). But then the analogy with turbulent advection in 2D would be lost.

VI. CONCLUDING REMARKS

The offered explanation of k^{-1} scaling is based entirely on consideration of vorticity filaments (thin stripes). It may seem surprising that the coherent vortices (CVs) that have attracted so much attention in the study of 2DT are allowed to contribute only indirectly, if at all, by sustaining, in the gaps between themselves, hyperbolic flow regions where filaments are stretched. In Kraichnan's theory large-scale strains, and thus CVs, are not required. The DNS discussed above (see also [4, Fig. 3]) showed that CVs may not dominate the flow, whether $\beta = 0$ or not, and a k^{-1} scaling will still hold under restrictions (1) and (2). Even if $\beta = 0$, the CV contribution to the spectrum for wave numbers above the energy-containing range is negligible: Ohkitani [19] showed that the total contribution from regions with elliptic flow, including all CVs, scales as k^{-2}.

The present article claims the existence of statistically steady states with k^{-1} enstrophy scaling over the full range of wave numbers. In particular, the enstrophy cascade should be sustainable by a small-scale forcing, which seems surprising. This implies the presence of an inverse cascade of energy from the forcing range toward the largest scales which are only damped linearly ($\nu_0 > 0$). But this cascade is not allowed to show up in spectra or in averaged nonlinear transfer functions, i.e., the usually observed $k^{-5/3}$ scaling is totally overwhelmed by filamentation when forcing is sufficiently weaker than the energy already present in the flow. Such dynamics seems possible only if the inverse energy and forward enstrophy cascades are ''additive,'' or mutually ''transparent,'' i.e., the responsible dynamical mechanisms are different and essentially *linear* for both cascades.

The k^{-3} mechanism proposed here is clearly linear. ''Transparency'' has already been demonstrated numerically by Maltrud and Vallis [4, Sec. 6]. The possibility of an inverse cascade of energy taking place along with a forward cascade of enstrophy was also verified by Smith and Yakhot [20], who were concerned with the inverse energy cascade in the first place. Although their large-scale energy spectrum is slightly shallower than k^{-3}, their Fig. 13 shows half a decade of ''enstrophy cascade'' at the lowest k's. Still lacking is a theoretical model of the ''agent'' of inverse energy cascade, which should be linear but should not require a $k^{-5/3}$ scaling for its existence. In a numerical experiment set up to investigate the inverse energy cascade, Borue [21] has shown that while a $k^{-5/3}$ scaling holds for the ''background vorticity'' that presumably effects that cascade, a steeper spectrum k^{-p} with $2 < p < 3$ dominates the low wave numbers. He showed that the $k^{-5/3}$ scaling pertains to the ''background'' component even at low k's, which strongly corroborates the present argument that the energy and enstrophy cascades must be linearly independent. In view of the analysis presented here, his argument that the nearly k^{-3} scaling of the overall spectrum, which he observed at low k's, is due to the presence of coherent structures is seen to be specific to the isotropic setup. It seems that a linear superposition of spectra, implicit already in the Maltrud and Vallis derivation [14] of the cross-over wave number between k^{-3} and $k^{-5/3}$ scaling, should result whenever there are sources of both filamentation and sufficiently strong (as in [21]) and sufficiently weakly correlated small-scale energy input.

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